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Two-Dimensional Field Theories Close to Criticality

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Using techniques developed in the context of conformal invariant two-dimensional field theories, we present some examples of finite-size-effect calculations in the vicinity of the critical point perturbed by relevant operators.

KEY WORDS: Conformal invariance; vicinity of the critical point; perturbation by relevant operators.

1. INTRODUCTION

Applications of conformal invariance, i.e., local scale invariance, to two-dimensional statistical physics have already been numerous and delt mostly with properties at the critical point. It is a natural idea to try to exploit some of the new techniques that have been developed in this context to study the vicinity of the critical point, where continuous field theory applies. This can be called the critical domain. Although there has been a lot of work of this kind, mostly in the case of integrable models, we know of only two recent attempts using the machinery of conformal invariance. One is by Cardy,[(1)] who studies the logarithmic corrections to finite-size scaling due to marginally irrelevant operators, and the other by Zamolodchikov,[(2)] who generalizes the short-distance algebra to the situation where the trace of the energy-momentum tensor does not vanish and a new scale is present.

Our goal is to present a few calculations and examples of what is to be naturally expected. Due to the appearance of a length scale, it is tempting to relate it to a particular geometry. The simplest one obtained from periodic boundary conditions leads to a torus and has been very useful in previous studies. It provides, furthermore, the constraint of modular

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invariance, the latter being by no means restricted to massless field theories. Even though this may not be as powerful in the general case, it still allows useful checks. Moreover, any theory in a finite box delivers us from infrared singularities if one attempts to develop a perturbation theory around criticality.

The deviations from criticality can be interpreted as additions to the Hamiltonian of terms coupled to relevant operators, i.e., of dimension less than or equal to two. Those lead in particular to an ultraviolet finite perturbation theory, a second fortunate circumstance, which is further enhanced by the fact that the latter is most likely a convergent one.

Turning our attention to the observables, one may inquire which are those of greatest interest. A conflicting demand is to allow for "easy" computation. We find it convenient in a first approach to study the partition function (on the torus) and its deformation as a function of the sources coupled to the relevant operators (for instance, deviation in temperature from T_c or analogs of a magnetic field). As is well known, this partition function at T_c codes for the set of critical indices as well as the central charge. It is therefore no surprise to see this structure evolve in such a way that we get quantities generalizing the central charge and critical indices becoming functions of the coupling parameters (the sources) and the length scale.

The paper opens with a section on free fields, where we compute the partition function for twisted boundary conditions in the presence of a mass term. This computation is similar to the massless case and results in Eq. (2.23). In Section 3 we apply our previous result to the Ising model (free massive fermions) and we recover expressions obtained nearly two decades ago by Ferdinand and Fisher[3] [Eq. (3.3)]. Modular invariance is a consequence of the derivation, but is explicit in the expansion of the free energy in inverse powers of the correlation length (proportional to $T - T_c$). Moreover, this a convergent expansion with a finite radius of convergence limited by the nearest singularity in the complex T plane (a zero of the partition function in the appropriate scale). An analysis of the formula suports the expectations. One obtains in closed form the scale-dependent central charge, and other dimensions. More interesting, a splitting appears between order and disorder operators, also as expected. Finally, the additive (ultraviolet) renormalization of the specific heat is observed.

Unfortunately, it is not known how to extend these analytic results to other cases. One is therefore led to study a perturbation expansion that relies on the knowledge of critical correlation functions in the *infinite* plane. The latter are in principle known. After presenting the general formalism, we restrict our attention to specific calculations in the Ising case. We recover some of the results of Section 2 by resumming the perturbation series for a thermal perturbation. We present computations up to fourth order in the magnetic field [Eqs. (4.14), (4.29)]. There are no obstacles in principle to continuing further except that the expressions would become extremely cumbersome. We are able to check these expressions against a number of data, including previous studies of finite-size effects for the ratio $\langle M^4\rangle/\langle M^2\rangle^2$ (M is the magnetization) in a periodic strip at criticality. The agreement is quite satisfactory. It would of course be most interesting to obtain a complete expression for the effective central charge, say, and see its evolution toward the Lee–Yang edge singularity in imaginary magnetic field, which gives the finite radius of convergence of the perturbative series in this case.

It is a challenge to find efficient methods (possibly nonperturbative) to perform similar calculations in the Ising and other models. But even accepting this limitation, our examples show that at least perturbative techniques work quite well and ought to be applied to more instances.

More generally and speculating even further, it has already been observed that there is a close connection between integrable two-dimensional models and the ones described by a finite number of primary fields at criticality. This connection remains very mysterious and leads to several questions: for instance, where is the Yang–Baxter algebra hidden and what are the relations between the statistical weights of integrable models and various partitions functions constructed at criticality which seem to use very similar ingredients? One may hope, perhaps a bit naively, that pursuit of the study away from T_c might shed some light on these questions if one attempts, for instance, to evaluate the S-matrix scattering elements. Related to this problem is the one of understanding the link between systems of equations for Green's functions at T_c derived from the short-distance algebra and the corresponding Painlevé systems which govern in the best of all cases the same correlations off T_c. We hope to return to these questions in the future.

2. FREE FIELDS. KRONECKER'S FORMULA

1. We derive in this section the partition function of a free, massive field subjected to twisted boundary conditions on a torus $\mathbb{T}$. In the Euclidean plane the torus is thought of as a quotient by the lattice $\mathbb{L}$ generated by two periods ω_1 and ω_2. We use a complex notation and assume an orientation such that the ratio

$$\tau = \omega_2/\omega_1 \tag{2.1}$$

has a positive imaginary part. We introduce complex fields $\varphi(x)$ satisfying (k, l, N integers, $0 \leqslant k, l < N$)

$$\varphi(x + n_1\omega_1 + n_2\omega_2) = \exp\left(2i\pi \frac{ln_1 - kn_2}{N}\right) \varphi(x) \tag{2.2}$$

(where of course φ is a function of x and $\bar{x}$). Over such complex fields we want to give a meaning to (i.e., renormalize) the partition function written as a functional integral

$$\begin{aligned} D_{k/N,l/N}^{-2}(\mathbb{T}; m) &= \int D\varphi\bar{\varphi} \exp\left[-\int_{\mathbb{T}} d^2x\, \bar{\varphi}(-\Delta + m^2)\, \varphi\right] \\ &= \prod_{n_1, n_2} \frac{1}{E_{n_1 n_2}} \end{aligned} \tag{2.3}$$

The product is over the eigenvalues $E_{n_1 n_2}$ of the operator $-\Delta + m^2$ subject to the periodicity conditions (2.2), i.e.,

$$E_{n_1 n_2} = \left(\frac{2\pi}{A}\right)^2 \left|\left(n_1 + \frac{k}{N}\right)\omega_1 + \left(n_2 + \frac{l}{N}\right)\omega_2\right|^2 + m^2, \qquad n_1, n_2 \in \mathbb{Z} \tag{2.4}$$

$$A = \text{area of } \mathbb{T} = |\omega_1|^2 \operatorname{Im} \tau$$

and is only formal. To give a precise (renormalized) definition, one introduces the function

$$G(s) = \sum_{n_1, n_2} \frac{1}{(E_{n_1 n_2})^s} \tag{2.5}$$

at first analytic for $\operatorname{Re} s > 1$, but which admits a meromorphic continuation, analytic as it turns out at $s = 0$. Thus we set

$$D_{k/N,l/N}(\mathbb{T}; m) = \exp -\tfrac{1}{2} G'(0) \tag{2.6}$$

To perform the calculation, we introduce the notations

$$t = m \frac{|\omega_1|}{2\pi} \tag{2.7}$$

$$G(s) = \left(\frac{A \operatorname{Im} \tau}{4\pi^2}\right)^s g(s) \tag{2.8}$$

$$g(s) = \sum_{n_2} \sum_{n_1} \frac{1}{\{[n_1 + k/N + (n_2 + l/N) \operatorname{Re} \tau]^2 + \operatorname{Im} \tau^2 [(n_2 + l/N)^2 + t^2]\}^s} \tag{2.9}$$

in such a way that

$$D_{k/N,l/N} = \exp -\left[\frac{1}{2} g(0) \ln \frac{A \operatorname{Im} \tau}{4\pi^2} + \frac{1}{2} g'(0)\right] \tag{2.10}$$

2. In order to evaluate (and analytically continue) $g(s)$, we follow closely the procedure of Ref. 4, very much as we did it in the massless case.[5]

In (2.9) the sum over the integer n_1 defines a periodic function (period 1) in the variable $k/N + (n_2 + l/N) \operatorname{Re} \tau$. One then writes its expansion in Fourier series. Set

$$a = k/N + (n_2 + l/N) \operatorname{Re} \tau, \qquad b = \operatorname{Im} \tau [(n_2 + l/N)^2 + t^2]^{1/2} > 0 \tag{2.11}$$

Then

$$\begin{aligned}
&\sum_{n_1} \frac{1}{[(n_1 + a)^2 + b^2]^s} \\
&\quad = \sum_{n_1} \exp(2i\pi n_1 a) \int_{-\infty}^{+\infty} dy \exp(-2i\pi n_1 y) \frac{1}{(y^2 + b^2)^s} \\
&\quad = \sum_{n_1} \exp(2i\pi n_1 a) \frac{1}{\Gamma(s)} \int_0^\infty \frac{du}{u} u^s \int_{-\infty}^{\infty} dy \exp[-u(y^2 + b^2) - 2i\pi n_1 y] \\
&\quad = \frac{\pi^{1/2}}{\Gamma(s)} \sum_{n_1} \exp(2i\pi n_1 a) \int_0^\infty \frac{du}{u} u^{s-1/2} \exp -\left(ub^2 + \frac{n_1^2 \pi^2}{u}\right)
\end{aligned}$$

We rescale the integration variable and split off the contribution of the term $n_1 = 0$, with the result that ($\sum_n'$ means sum over all n except 0)

$$\begin{aligned}
&\sum_{n_1} \frac{1}{[(n_1 + a)^2 + b^2]^s} \\
&\quad = \frac{\pi^{1/2}}{\Gamma(s)} \left\{ \frac{\Gamma(s - 1/2)}{b^{2s-1}} + \sum_{n_1}' \exp(2i\pi n_1 a) \left(\frac{\pi\, |n_1|}{b}\right)^{s-1/2} \right. \\
&\qquad \left. \times \int_0^\infty \frac{dv}{v} v^{s-1/2} \exp[-\pi\, |n_1|\, b(v + 1/v)] \right\}
\end{aligned} \tag{2.12}$$

We now reinstate the values for a and b given by (2.11) and sum over n_2 to get $g(s)$. The second term in (2.12) is readily evaluated. Taking the limit $s \to 0$ in the integral and using the fact that

$$\int_0^\infty \frac{dv}{v} v^{\pm 1/2} e^{-\lambda(v + 1/v)} = \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-2\lambda} \tag{2.13}$$

$$g(s)=\frac{\pi^{1/2}}{\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)\operatorname{Im}\tau^{1-2s}\sum_n\frac{1}{[(n+l/N)^2+t^2]^{s-1/2}}$$

$$-2s\ln\left|\prod_n\left(1-\exp\left\{2i\pi\left[\frac{k}{N}+\left(\frac{n+l}{N}\right)\operatorname{Re}\tau\right]\right.\right.\right. \tag{2.14}$$

$$\left.\left.\left.-2\pi\operatorname{Im}\tau\left[\left(\frac{n+l}{N}\right)^2+t^2\right]^{1/2}\right\}\right)\right|+O(s^2)$$

then we have to treat with some care the first term in (2.14), depending on l/N only, which we write as

$$\hat{g}_{l/N}(s)=\frac{\pi^{1/2}}{\Gamma(s)}\Gamma\left(s-\frac{1}{2}\right)\operatorname{Im}\tau^{1-2s}\sum_n\frac{1}{[(n+l/N)^2+t^2]^{s-1/2}} \tag{2.15}$$

Two cases are to be distinguished. If $l/N=0$

$$\hat{g}_0(s)=\frac{\pi\operatorname{Im}\tau^{1-2s}}{\Gamma(s)}\left[\frac{t^{1-2s}\Gamma(s-1/2)}{\pi^{1/2}}+\frac{2\Gamma(s-1/2)\,\zeta(2s-1)}{\pi^{1/2}}\right.$$

$$-\frac{2\Gamma(s+1/2)\,\zeta(2s+1)\,t^2}{\pi^{1/2}}$$

$$\left.+\frac{2\Gamma(s+3/2)\,t^4}{\pi^{1/2}}\int_0^1 d\lambda(1-\lambda)\sum_{n=1}^{\infty}\frac{1}{(n^2+\lambda t^2)^{3/2}}\right]$$

Taking into account that

$$\zeta(-1)=-1/12,\qquad \zeta(1+2s)=1/2s(1+2s\gamma+\cdots)$$
$$\Gamma(s+1/2)=\sqrt{\pi}\,[1-s(\gamma+2\ln 2)+\cdots]$$
$$\Gamma(-1/2)=-2\sqrt{\pi},\qquad 1/\Gamma(s)=s(1+\gamma s+\cdots)$$

were γ is Euler's constant, we find

$$\hat{g}_0(s)=\pi\operatorname{Im}\tau\left\{-t^2+s\left[\frac{1}{3}-2t+2t^2\ln(2\operatorname{Im}\tau e^{-\gamma})\right.\right.$$

$$\left.\left.+t^4\int_0^1 d\lambda\sum_{n=1}^{\infty}\frac{1-\lambda}{(n^2+\lambda t^2)^{3/2}}\right]\right\}+O(s^2) \tag{2.16}$$

Similarly,

$$\hat{g}_{l/N}(s) = \frac{\pi \operatorname{Im} \tau^{1-2s}}{\Gamma(s)} \left\{ \frac{\Gamma(s-1/2)}{\pi^{1/2}} \sum_n \frac{1}{|n+l/N|^{2s-1}} \right.$$
$$- \frac{t^2 \Gamma(s+1/2)}{\pi^{1/2}} \sum_n \frac{1}{|n+l/N|^{2s+1}}$$
$$\left. + \frac{\Gamma(s+3/2)\, t^4}{\pi^{1/2}} \int_0^1 d\lambda \sum_n \frac{1-\lambda}{[(n+l/N)^2+\lambda t^2]^{s+3/2}} \right\} \tag{2.17}$$

Except for the middle term, the limit $s \to 0$ is easily taken, since the first and last terms in the bracket have finite limits as $s \to 0$. In particular,

$$\lim_{s \to 0} \frac{\Gamma(s-1/2)}{\pi^{1/2}} \sum_n \frac{1}{|n+l/N|^{2s-1}} = \frac{1}{\pi^2} \sum_n \frac{e^{2i\pi n l/N}}{n^2}$$
$$= \frac{1}{3}\left[1 - \frac{6l(N-l)}{n^2}\right]$$

Thus,

$$\hat{g}_{l/N}(s) = \pi \operatorname{Im} \tau\, s \left\{ \frac{1}{3}\left[1 - \frac{6l(N-l)}{N^2}\right] \right.$$
$$\left. + \frac{t^4}{2} \int_0^1 d\lambda \sum_n \frac{1-\lambda}{[(n+l/N)^2+\lambda t^2]^{3/2}} \right\}$$
$$- \pi \operatorname{Im} \tau\, t^2 \frac{\operatorname{Im} \tau^{-2s}\, \Gamma(s+1/2)}{\pi^{1/2}\Gamma(s)} \sum_n \frac{1}{|n+l/N|^{2s+1}} + O(s^2) \tag{2.18}$$

Consider now

$$\frac{\Gamma(s+1/2)}{\pi^{1/2}\Gamma(s)} \sum_n \frac{1}{|n+l/N|^{2s+1}}$$
$$= \frac{\Gamma(s+1/2)}{\pi^{1/2}\Gamma(s)} \left[H\left(\frac{l}{N}; 1+2s\right) + H\left(1-\frac{l}{N}; 1+2s\right) \right] \tag{2.19}$$

where

$$H(x, \sigma) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^\sigma}$$

If $\sigma \to 1$,

$$H(x, \sigma) \simeq \frac{1}{\sigma - 1} + \alpha(x) + \cdots \tag{2.20}$$

where α has to be determined. One has

$$\frac{\partial}{\partial x} H(x, \sigma) = -\sigma H(x, \sigma + 1)$$

$$\frac{\partial^2 H(x, \sigma)}{\partial x\, \partial \sigma} = -H(x, \sigma + 1) - \sigma \frac{\partial}{\partial \sigma} H(x, \sigma + 1)$$

Let $\sigma \to 0$; then

$$\frac{\partial^2 H(x, \sigma)}{\partial x\, \partial \sigma} = -\alpha(x) + \cdots$$

and since one has

$$\left.\frac{\partial}{\partial \sigma} H(x, \sigma)\right|_{\sigma = 0} = \ln \frac{\Gamma(x)}{(2\pi)^{1/2}}$$

(Lerch formula[4]), we get

$$\alpha(x) = -\psi(x) \tag{2.21}$$

where ψ is the logarithmic derivative of Euler's gamma function. Thus, finally,

$$\begin{aligned} \hat{g}_{l/N}(s) = \pi \operatorname{Im} \tau \Bigg\{ -t^2 + s \Bigg[\frac{1}{3}\left(1 - \frac{6l(N-l)}{N^2}\right) \\ + t^2 \left[2 \ln \operatorname{Im} \tau + \psi\left(\frac{l}{N}\right) + \psi\left(1 - \frac{l}{N}\right) + 2 \log 2 \right] \\ + \frac{t^4}{2} \int_0^1 d\lambda \sum_n \frac{1 - \lambda}{[(n + l/N)^2 + \lambda t^2]^{3/2}} \Bigg] \Bigg\} + O(s^2) \end{aligned} \tag{2.22}$$

Collecting our results, we obtain

$$\begin{aligned} D_{k/N, l/N}(\mathbb{T}, m) = \exp(-\pi \operatorname{Im} \tau \gamma_{l/N}) \\ \times \left| \prod_n \left(1 - \exp\left\{ 2i\pi \left[\frac{k}{N} + \left(n + \frac{l}{N}\right) \operatorname{Re} \tau \right] \right.\right.\right. \\ \left.\left.\left. - 2\pi \operatorname{Im} \tau \left[\left(n + \frac{l}{N}\right)^2 + t^2 \right]^{1/2} \right\} \right) \right| \end{aligned} \tag{2.23}$$

where

$$\gamma_0 = \frac{1}{6} - t + t^2 \ln\left[4\pi e^{-\gamma}\left(\frac{\operatorname{Im}\tau}{A}\right)^{1/2}\right]$$
$$+ \frac{t^4}{2}\int_0^1 d\lambda(1-\lambda)\sum_{n=1}^{\infty}\frac{1}{(n^2+\lambda t^2)^{3/2}}$$
$$\gamma_{l/N(l/N\neq 0)} = \frac{1}{6}\left[1 - \frac{6l(N-l)}{N^2}\right]$$
$$+ t^2\left[\ln 4\pi\left(\frac{\operatorname{Im}\tau}{A}\right)^{1/2} + \frac{1}{2}\psi\left(\frac{l}{N}\right) + \frac{1}{2}\psi\left(1-\frac{l}{N}\right)\right]$$
$$+ \frac{t^4}{4}\int_0^1 d\lambda(1-\lambda)\sum_n \frac{1}{[(n+l/N)^2+\lambda t^2]^{3/2}}$$

In particular,

$$\gamma_{1/2} = -\frac{1}{12} + t^2 \ln\left[\pi e^{-\gamma}\left(\frac{\operatorname{Im}\tau}{A}\right)^{1/2}\right]$$
$$+ \frac{t^4}{2}\int_0^1 d\lambda(1-\lambda)\sum_{n=0}^{+\infty}\frac{1}{[(n+1/2)^2+\lambda t^2]^{3/2}}$$

We note that $\gamma_{l/N} - \gamma_0$ is a function of t only.

3. As a first application of the above resulrs, we investigate the partition function of a scalar, periodic, real field off-criticality, i.e., at $m \neq 0$. From (2.3) it is given by

$$Z_{\text{scalar}} = \frac{1}{D_{00}}$$
$$= \exp \pi \operatorname{Im}\tau \left\{\frac{1}{6} - t + t^2 \ln\left[4\pi \exp(-\gamma)\left(\frac{\operatorname{Im}\tau}{A}\right)^{1/2}\right]\right.$$
$$\left. + \frac{t^4}{2}\int_0^1 d\lambda(1-\lambda)\sum_{n=1}^{\infty}\frac{1}{(n^2+\lambda t^2)^{3/2}}\right\}$$
$$\times \prod_n \{1 - \exp[2i\pi n \operatorname{Re}\tau - 2\pi \operatorname{Im}\tau(n^2+t^2)^{1/2}]\}^{-1} \quad (2.24)$$

The infinite product is of course real and positive, so no absolute value sign is necessary. We recall that $t = m\,|\omega_1|/2\pi$ is assumed to be positive in this expression.

The interpretation of the formula is clear. From the derivation, it is a modular invariant, even though it is expressed using variables adapted to a specific basis. In particular, t is a dimensionless measure of the departure from criticality defined in terms of specific period. One can interpret the prefactor as $\exp \pi/6 \operatorname{Im} \tau\, C(t, \operatorname{Im} \tau)$, where $C(0, \operatorname{Im} \tau) = 1$ as a varying or scale-dependent "central charge." The appearance of a $\ln \operatorname{Im} \tau$ dependence of the t^2 coefficient reflects the necessity of an ultraviolet renormalization of the "specific heat," i.e., the second derivative of $\ln Z$ with respect to m. The infinite product in the denominator could have been easily predicted in a transfer-matrix formalism by decomposing the field in proper modes. The discrete momenta $2\pi n/|\omega_1|$ are associated to eigenfrequencies $[(2\pi n/|\omega_1|)^2 + m^2]^{1/2}$, which is what is indicated in (2.24). By extracting the contribution of the zero mode responsible for a potential divergence, one easily recovers the massless result

$$\lim_{m \to 0} m \sqrt{A}\, Z(m) = \frac{1}{(\operatorname{Im} \tau)^{1/2}\, \eta(\tau)\, \bar{\eta}(\tau)} \tag{2.25}$$

where

$$\eta(\tau) = q^{1/24} \prod_1^\infty (1 - q^n), \qquad q = e^{2i\pi\tau}$$

3. ISING MODEL CLOSE TO T_c

In this section we study the Ising model in the scaling (or continuous field-theoretic) regime at a temperature slightly off criticality. The correlation length ξ is finite and $\xi^{-1} = |m|$, where m is proportional to $a^{-1}(T/T_c - 1)$, with a standing for a typical lattice spacing. The model is described by a path integral in terms of Grassmanian variables,

$$\begin{aligned} Z_{\text{Ising}} &= \int \mathscr{D}\psi\bar{\psi} \exp\left(2 \int_{\mathbb{T}} d^2z\, \psi\, \bar{\partial}\psi - \bar{\psi}\, \partial\bar{\psi} + m\bar{\psi}\psi\right) \\ &= [\operatorname{Det}(-\Delta + m^2)]^{1/2} \end{aligned} \tag{3.1}$$

There are several qualifications here. First, we have to figure out the boundary conditions on the torus. We assume that both ψ and $\bar{\psi}$ are either periodic or antiperiodic and sum over the *four* contributions. Second, in a discrete version on a finite domain the sign of $m \sim (T/T_c - 1)$ matters, as the path integral is polynomial in m. We perform the calculation with m positive, say, and then analytically continue through $m = 0$. As we shall see, a simple term is sensitive to the sign of m and one has to distinguish the disordered phase $T > T_c$ from the ordered one $T < T_c$. With this understanding we obtain

$$Z_{\text{Ising}}(m) = D_{1/2,1/2}(m) + D_{0,1/2}(m) + D_{1/2,0}(m) + D_{0,0}(m) \tag{3.2}$$

i.e.,

$$\begin{aligned}
Z_{\text{Ising}}(m) = \exp \pi \operatorname{Im} \tau \Bigg\{ & \frac{1}{12} - t^2 \ln \left[\pi \exp(-\gamma) \left(\frac{\operatorname{Im} \tau}{A} \right)^{1/2} \right] \\
& - \frac{t^4}{2} \int_0^1 d\lambda (1-\lambda) \sum_{n=1}^{\infty} \frac{1}{[(n-1/2)^2 + \lambda t^2]^{3/2}} \Bigg\} \\
& \times \Bigg[\sum_{\pm} \prod_n \Bigg(1 \pm \exp \Bigg\{ 2i\pi \left(n + \frac{1}{2} \right) \operatorname{Re} \tau \\
& - 2\pi \operatorname{Im} \tau \left[\left(n + \frac{1}{2} \right)^2 + t^2 \right]^{1/2} \Bigg\} \Bigg) \\
& + \exp - \pi \operatorname{Im} \tau \left\{ \frac{1}{4} - t + t^2 \ln 4 - \frac{t^4}{2} \int_0^1 d\lambda (1-\lambda) \right. \\
& \left. \times \sum_{n=1}^{\infty} \left(\frac{1}{[(n-1/2)^2 + \lambda t^2]^{3/2}} - \frac{1}{(n^2 + \lambda t^2)^{3/2}} \right) \right\} \\
& \times \sum_{\pm} \prod_n \{ 1 \pm \exp[2i\pi n \operatorname{Re} \tau - 2\pi \operatorname{Im} \tau (n^2 + t^2)^{1/2}] \} \Bigg]
\end{aligned} \tag{3.3}$$

This is the result obtained more than 15 years ago by Ferdinand and Fisher,[3] except perhaps for minor discrepancies due to misprints. [The term proportional to t^2 in the first exponential has been omitted in formula (3.36) of this paper, although it comes out simply from (3.30) and (3.32).] Absolute value signs have been suppressed, since products in n from $-\infty$ to $+\infty$ imply reality of the corresponding expressions. In the limit $m = 0$ we recover the expected results with $D_{00}(0)$ vanishing,

$$Z_{\text{Ising}}(0) = D_{1/2,1/2}(0) + D_{0,1/2}(0) + D_{1/2,0}(0)$$

$$\begin{aligned}
D_{1/2,1/2}(0) &= \left| q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}) \right|^2 \\
D_{0,1/2}(0) &= \left| q^{-1/48} \prod_{n=0}^{\infty} (1 - q^{n+1/2}) \right|^2 \\
D_{1/2,0}(0) &= 2 \left| q^{-1/48+1/16} \prod_{n=1}^{\infty} (1 + q^n) \right|^2
\end{aligned} \tag{3.4}$$

Our normalization differs here by an overall factor 1/2 (put in by hand) from the corresponding expressions given in Ref. 5. In (3.3) the terms sensitive to the change $t \leftrightarrow -t$ $(m \leftrightarrow -m)$ are in the exponential prefactor of $D_{1/2,0}$ and D_{00} and in their factor corresponding to $n=0$ written as $1 \pm \exp(-2\pi \operatorname{Im} \tau t)$. Combining these factors yield $\exp(\pi \operatorname{Im} \tau t) \pm \exp(-\pi \operatorname{Im} \tau t)$.

The relativistic massive spectrum of states occurring in (3.3) is quite natural, given that the corresponding boundary conditions are very much as in the scalar case. Furthermore, not only does the central charge become scale dependent, but also the "dimensions" of the various operators. For instance, the spin operator splits into two operators which were indistinguishable at T_c: the spin operator proper and the dual disorder operator related by $t \leftrightarrow -t$ with scale-dependent "dimensions." The occurrence of a $t^2 \ln[(\operatorname{Im} \tau)/A]^{1/2}$ term in the "central charge" is once again a manifestation of the renormalization properties of the operator $\bar{\psi}\psi$ of dimension one. In a mass perturbation theory an ultraviolet-divergent subtraction is required to second-order, related to the appearance of a $m^2 \ln L/a$ term in the specific heat, as indicated below. The calculation using ζ-function regularization has by-passed the problem and provided a nonperturbative finite answer.

The partition function is a combination of two modular invariants, namely $D_{1/2,1/2}(m) + D_{0,1/2}(m) + D_{1/2,0}(m)$ and $D_{00}(m)$. The latter is of course the inverse of the partition function of a scalar field and appears here as a combination of the spin and disorder operator sectors. It is instructive to expand the free energy $\ln Z(m)$ in powers of m. We shall use the notations D_{ij} instead of $D_{ij}(0)$ for the three quantities given in (3.4) and $Z_{1/2}$ for $Z_{\text{Ising}}(0)$. Furthermore, Z_1 will stand for the critical scalar partition function defined in (2.25),

$$Z_1 = \frac{1}{(\operatorname{Im} \tau)^{1/2}\, \eta(\tau)\, \bar{\eta}(\tau)}$$

Then, expanding (3.3) to order m^2, one finds after some calculations

$$\begin{aligned} \frac{Z_{\text{Ising}}(m)}{Z_{1/2}} = 1 &+ \frac{mA^{1/2}}{Z_1 Z_{1/2}} + \frac{m^2 A}{4\pi} \left[\ln\left(\frac{Z_1 \sqrt{A}\, e^{\gamma}}{\pi} \right) - \frac{2 \sum D_{ij} \ln D_{ij}}{Z_{1/2}} \right] \\ &+ O[(mA^{1/2})^3] \end{aligned} \tag{3.5}$$

where the sum runs over the three combinations of indices (1/2, 1/2), (0, 1/2), and (1/2, 0) and can be thought of as an entropic contribution of the various spin structures. Equivalently,

$$
\begin{aligned}
\ln Z_{\text{Ising}}(m) = \ln Z_{1/2} &+ \frac{mA^{1/2}}{Z_1 Z_{1/2}} \\
&+ m^2 A \left[\frac{1}{4\pi} \ln \left(\frac{Z_1 \sqrt{A}\, e^{\gamma}}{\pi} \right) - \frac{1}{2\pi} \frac{\sum D_{ij} \ln D_{ij}}{Z_{1/2}} - \frac{1}{2(Z_1 Z_{1/2})^2} \right] \\
&+ O[(mA^{1/2})^3]
\end{aligned} \tag{3.6}
$$

a rather neat expression involving both the fermionic ($Z_{1/2}$) as well as bosonic (Z_1) partition functions.

One notes the appearance of a term linear in m corresponding to a nonzero expectation value of the energy operator in the spin sector (and not the "vacuum" one). We shall discuss below the details of this mechanism. The arbitrary length scale involved in the term $m^2 \ln \sqrt{A}$ appears as an additive renormalization of the specific heat. If we return to the original starting point [Eq. (2.3)], we see that the product over energies is dimensional. We could have *a priori* decided to measure lengths in units of $\sqrt{A}$ and energies in units A^{-1}, which would amount to computing, instead of (2.3), the dimensionless quantity $\prod_{n_1 n_2} (AE_{n_1 n_2})^{-1}$ and would result in the effective disappearance of the factor $\sqrt{A}$ inside the logarithms of (3.6).

The existence of odd as well as even terms in the expansion (3.6) shows that in general the maximum of the specific heat is not at $m = 0$ ($T = T_c$), but depends on the modular ratio as observed in Ref. 3. Finally, let us note that in spite of the original disymmetric appearance, the expansion (3.6) is a testimony to modular invariance, involving only such quantities as Z_1, $Z_{1/2}$, m, A,....

4. PERTURBATION THEORY

1. Assuming the geometry depicted on Fig. 1a, we write the partition function of a continuous field-theoretic model as

$$
Z = \operatorname{Tr} \exp(-\operatorname{Im} \omega_2\, H + iP \operatorname{Re} \omega_2) \tag{4.1}
$$

where H and P are the commuting Hamiltonian and momentum operators. For definiteness we use a coordinate system in the u plane where ω_1 is real (Fig. 1). The trace is of course independent of the choice of basis in which we have a realization of H and P. In any case we choose one in which P is diagonal. At criticality, H reduces to H_0,

$$
H_0 = \frac{2\pi}{\omega_1} \left(L_0 + \bar{L}_0 - \frac{C}{12} \right) \tag{4.2}
$$

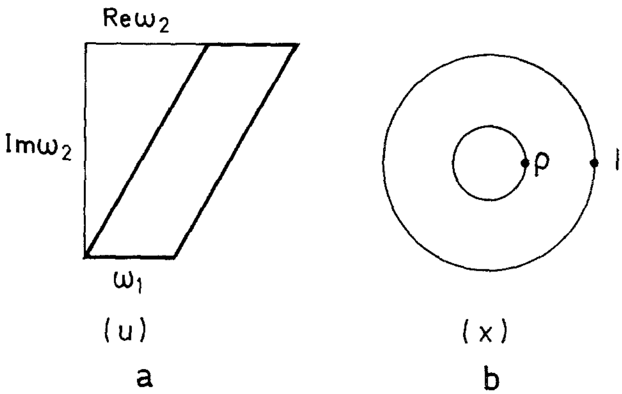

Fig. 1

with L_0 and $\bar{L}_0$ belonging to the commuting Virasoro algebras[6] generated by the energy-momentum operator and C is the central charge. The states form a (decomposable) representation of these algebras. We shall assume the vicinity of the critical theory described by a modified Hamiltonian

$$H = H_0 + V = H_0 - \sum G \int_0^{\omega_1} \frac{du_1}{2\pi} \varphi(u_1, 0) \tag{4.3}$$

with a sum over a set of primary local real fields of conformal weights $(h, \bar{h} = h)$. To simplify notations we limit ourselves here to a single field and suppress the summation sign. This means that the coupling constant G is of dimension $2 - 2h$ in units of $[\text{length}]^{-1}$.

To start with, we would like to understand the "running" central charge (a scale-dependent quantity), the analog of the phenomenon found in the previous section. For that purpose instead of computing the full trace in (4.1), we limit ourselves first to the unperturbed vacuum contribution. We denote by $|0\rangle$ this translationally invariant state and write

$$\begin{aligned} Z_0 &= \langle 0 | \exp(-\operatorname{Im} \omega_2 H + iP \operatorname{Re} \omega_2) | 0 \rangle \\ &= \exp \frac{\pi \operatorname{Im} \tau C}{6} \langle 0 | \mathcal{T} \exp \frac{G}{2\pi} \int_0^{\operatorname{Im} \omega_2} du_2 \int_0^{\omega_1} du_1 \, \varphi(u_1, u_2) | 0 \rangle \end{aligned} \tag{4.4}$$

where the "time" ordering operator $\mathcal{T}$ acts on the variable u_2.

It is then useful to return to the punctured plane (Fig. 1b) using the map

$$x = \exp \frac{2i\pi}{\omega_1} u, \qquad u = u_1 + iu_2 \tag{4.5}$$

Thus,

$$d^2u = \left(\frac{\omega_1}{2\pi}\right)^2 \frac{d^2x}{x\bar{x}}$$

and from the conformal transformation properties of φ

$$d^2u\, \varphi_{\text{torus}}(u, \bar{u}) = \left(\frac{\omega_1}{2\pi}\right)^{2-2h} \frac{d^2x}{(x\bar{x})^{1-h}}\, \varphi_{\text{plane}}(x, \bar{x}) \tag{4.6}$$

The integral becomes one in an annulus from $|x| = \rho$ to $|x| = 1$ and ordering is radial. The value of ρ is

$$\rho = \exp - 2\pi \operatorname{Im} \tau \tag{4.7}$$

It is now useful to define a dimensionless coupling

$$g = G(|\omega_1|/2\pi)^{2-2h} \tag{4.8}$$

The case treated in Section 3 corresponded to the Ising model with $G = m$, $h = 1/2$ and g was denoted t. On the other hand, we could consider the case of a magnetic perturbation again in the Ising model with $G \to H$ (the magnetic field), $\varphi \to$ spin field of dimension $h = 1/16$, in which case $g = H(|\omega_1|/2\pi)^{15/8}$. Relevant perturbations will be such that $2 - 2h > 0$, in which case for fixed G, g grows as $|\omega_1| \to \infty$ and perturbation theory is dangerous unless G also goes to zero to maintain a finite (but small) g. On the other hand, this is also the most interesting one, as it may transfer one from a universal class to another. Two strategies are possible: One can study models such that one has an operator φ with h close to unity and develop renormalization group equations by expansion in $(1-h)$. This is a case studied by Zamalodchikov.[(2)] Alternatively, one can look at a straightforward expansion in g. This is what will be attempted here for relevant operators. For $2h$ a fractional number between 0 and 1, as the infrared cutoff is realized by the choice of boundary conditions and the ultraviolet behavior under control, no singularities appears in the g-expansion of Z_0 written as

$$\begin{aligned} Z_0(g, \operatorname{Im} \tau) &= \exp \frac{\pi \operatorname{Im} \tau C}{6} \langle 0|\, \mathscr{T} \exp \frac{g}{2\pi} \int_\rho^1 \frac{d^2x}{(x\bar{x})^{1-h}}\, \varphi(x, \bar{x})\, |0\rangle \\ &= \exp \frac{\pi \operatorname{Im} \tau C}{6} \sum_{N=0}^{\infty} \left(\frac{g}{2\pi}\right)^N \int_{\rho \leqslant |x_1| \cdots \leqslant |x_{2N}| \leqslant 1} \frac{d^2x_1 \cdots d^2x_N}{(x_1\bar{x}_1 \cdots x_N\bar{x}_N)^{1-h}} \\ &\quad \times \langle 0|\, \varphi(x_1, \bar{x}_1) \cdots \varphi(x_N, \bar{x}_N)\, |0\rangle \end{aligned} \tag{4.9}$$

If H has a ground state for $g \neq 0$, we presume that $\ln Z_0$ will behave like $\operatorname{Im} \tau$ in the limit $\operatorname{Im} \tau \to \infty$. This justifies the definition

$$C(g) = \lim_{\operatorname{Im} \tau \to \infty} \frac{\ln Z_0(g, \operatorname{Im} \tau)}{2\pi \operatorname{Im} \tau} = \lim_{\rho \to 0} \frac{\ln Z_0(g, \rho)}{\ln(1/\rho)} \tag{4.10}$$

with

$$C(g) = C/12 + g^2 C_2 + g^4 C_4 + \cdots \tag{4.11}$$

(assuming that odd correlation functions vanish) and

$$\begin{aligned} C_{2N} = \frac{1}{(2\pi)^{2N}} \lim_{\rho \to 0} \frac{1}{\ln(1/\rho)} \int_{\rho \leqslant |x_1| \cdots x_{2N}| \leqslant 1} \frac{d^2x_1 \cdots d^2x_{2N}}{(x_1 \bar{x}_1 \cdots x_{2N} \bar{x}_{2N})^{1-h}} \\ \times \langle 0 | \, \varphi(x_1, \bar{x}_1) \cdots \varphi(x_{2N}, \bar{x}_{2N}) \, | 0 \rangle_{\text{connected}} \end{aligned} \tag{4.12}$$

The necessary ingredient in the computation is provided by a knowledge of the critical correlation functions of primary fields. Those are in principle determined by the differential equations derived by Belavin *et al.*[6] and elaborated by Dotsenko and Fateev.[7] The overall normalization will be provided by the two-point function written as

$$\langle 0 | \, \varphi(x_1, \bar{x}_1) \, \varphi(x_2, \bar{x}_2) \, | 0 \rangle = 1/|x_1 - x_2|^{4h} \tag{4.13}$$

which then gives an absolute meaning to the parameter g. The one-point expectation value is assumed to vanish (for $h \neq 0$ in the infinite plane). The value of C_2 is then easily extracted. We have

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int_\rho^1 \frac{d^2x_1}{|x_1|^{2-2h}} \int_{|x_1|}^1 \frac{d^2x_2}{|x_2|^{2-2h}} \frac{1}{|x_1 - x_2|^{4h}} \\ &= \int_\rho^1 \frac{dx_2}{|x_2|^{1+2h}} \int_\rho^{x_2} \frac{dx_1}{|x_1|^{1-2h}} \\ &\quad \times \int_0^{2\pi} \frac{d\theta}{2\pi} \left(1 - \frac{x_1 e^{i\theta}}{x_2}\right)^{-2h} \left(1 - \frac{x_1 e^{-i\theta}}{x_2}\right)^{-2h} \\ &= \int_\rho^1 \frac{dx_2}{|x_2|^{1+2h}} \int_\rho^{x_2} \frac{dx_1}{|x_1|^{1-2h}} \sum_{n=0}^{\infty} \left(\frac{x_1}{x_2}\right)^{2n} \left[\frac{\Gamma(n+2h)}{n! \, \Gamma(2h)}\right]^2 \\ &= \sum_{n=0}^{\infty} \frac{1}{2n+2h} \left[\frac{\Gamma(n+2h)}{n! \, \Gamma(2h)}\right]^2 \int_\rho^1 \frac{dx_2}{x_2} \left[1 - \left(\frac{\rho}{x_2}\right)^{2n+2h}\right] \end{aligned}$$

Taking the limit $\rho \to 0$ in the above expression yields

$$
\begin{aligned}
C_2(h) &= \sum_0^{\infty} \frac{1}{2n+2h} \left[\frac{\Gamma(n+2h)}{n!\,\Gamma(2h)} \right]^2 \\
&= \frac{1}{2} \int_0^1 \frac{dx}{x}\, x^h F(2h, 2h; 1; x)
\end{aligned} \tag{4.14}
$$

The above series is of course meaningful provided

$$0 < 2h < 1$$

as stated above. An example of the limiting case $2h \to 1$ corresponds to the thermal operator in the Ising case treated in Section 3. The behavior of $\ln Z_0$ is no longer linear in $\operatorname{Im} \tau$ and a logarithmic divergence occurs in C_2 (see above). One could as well obtain (4.14) by a standard perturbation theory for the ground state of H_0, the denominators $2n+2h$ corresponding to the difference between the energy of excited states $|h+n, h+n\rangle$ coupled to $|0\rangle$ by the perturbation, and the numerator to the square modulus of matrix elements:

$$C(g) = \frac{C}{12} + \left(\frac{|\omega_1|}{2\pi} \right)^2 \sum \frac{|\langle 0|\, V\, |h+n, h+n\rangle|^2}{2n+2h} + \cdots \tag{4.15}$$

Sum rules necessary to the evaluation of (4.15) are provided by transformation of (4.13) and lead to calculations similar to the previous ones. Computation of the other terms requires a knowledge of multipoint correlation functions, which depend on the model.

2. Rather than trying to pursue the matter in general, we shall concentrate first on the thermal perturbation in the Ising case. Comparison with the exact results of the previous section will provide a check of the method. Here φ is the energy operator $\mathscr{E}$ with dimensions (1/2, 1/2). Odd correlation functions vanish, and even ones are given by the simple formula derived from Wick's theorem for the free field Fermi theory,

$$\langle 0|\, \mathscr{E}(x_1, \bar{x}_1) \cdots \mathscr{E}(x_{2N}, \bar{x}_{2N})\, |0\rangle = \left| Pf \frac{1}{x_{ij}} \right|^2 \tag{4.16}$$

Consider first the fourth-order term in (4.11). One has

$$\langle 0|\, \mathscr{E}(1) \cdots \mathscr{E}(4)\, |0\rangle = \left| \frac{1}{x_{12}x_{34}} - \frac{1}{x_{13}x_{24}} + \frac{1}{x_{14}x_{23}} \right|^2 \tag{4.17}$$

and the four-point connected correlation function reads

$$\langle 0|\,\mathscr{E}(1)\cdots\mathscr{E}(4)\,|0\rangle_{\text{connected}}$$
$$=\frac{1}{x_{12}x_{34}\bar{x}_{14}\bar{x}_{23}}-\frac{1}{x_{12}x_{34}\bar{x}_{13}\bar{x}_{24}}-\frac{1}{x_{13}x_{24}\bar{x}_{14}\bar{x}_{23}}+\text{C.C.}$$

A term like $(x_{12}x_{34}\bar{x}_{14}\bar{x}_{23})^{-1}$ can be rewritten

$$[x_2\bar{x}_3x_4\bar{x}_4(1-x_1/x_2)(1-x_3/x_4)(1-\bar{x}_1/\bar{x}_4)(1-\bar{x}_2/\bar{x}_3)]^{-1}$$

The $x_2\bar{x}_3$ in this expression produces a phase factor $\exp(-i\theta_{23})$ which will not be compensated by the expansion of the other factors, leading to a vanishing contribution after the angular integration in (4.12). We are thus left with

$$-\frac{1}{(2\pi)^4}\int_{\rho\leqslant|x_1|\cdots\leqslant|x_4|\leqslant 1}\frac{d^2x_1\cdots d^2x_4}{(x_1\bar{x}_1\cdots x_4\bar{x}_4)^{1/2}}$$
$$\times\left[x_3\bar{x}_3x_4\bar{x}_4\left(1-\frac{x_1}{x_3}\right)\left(1-\frac{x_2}{x_4}\right)\left(1-\frac{\bar{x}_1}{\bar{x}_4}\right)\left(1-\frac{\bar{x}_2}{\bar{x}_3}\right)\right]^{-1}+\text{C.C.}$$
$$=-\int_\rho^1\frac{dx_4}{x_4^2}\int_\rho^{x_4}\frac{dx_3}{x_3^2}\int_\rho^{x_3}dx_2\int_\rho^{x_2}dx_1$$
$$\times\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{d\theta_1\cdots d\theta_4}{(2\pi)^4}\left[\left(1-\frac{x_1e^{i\theta_{13}}}{x_3}\right)\left(1-\frac{x_2e^{i\theta_{24}}}{x_4}\right)\right.$$
$$\left.\times\left(1-\frac{x_1}{x_4}e^{i\theta_{14}}\right)\left(1-\frac{x_2}{x_3}e^{i\theta_{23}}\right)\right]^{-1}+\text{C.C.}$$
$$=-2\int_\rho^1\frac{dx_4}{x_4^2}\int_\rho^{x_4}\frac{dx_3}{x_3^2}\int_\rho^{x_3}dx_2\int_\rho^{x_2}dx_1\sum_{n=0}^{\infty}\left(\frac{x_1x_2}{x_3x_4}\right)^{2n}$$

When ρ goes to zero we replace all the lower limits of integration by 0, except for the first one. We get then

$$-2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\,(4n+2)}\int_\rho^1\frac{dx_4}{x_4}$$

and thus

$$C_4(1/2)=-\zeta(3)(1-2^{-3})\tag{4.18}$$

In contrast to $C_2(1/2)$, this is finite, and agrees with (3.3). The following $C_{2N}(1/2)$ can be calculated exactly in the same way. The only terms in the connected $2N$ correlation function (4.16) that lead to a nonvanishing result after angular integration are of the form $(x_{1i_1}\cdots x_{Ni_N}\bar{x}_{1j_1}\cdots\bar{x}_{Nj_N})^{-1}$ where $(i_1\cdots i_N)$ and $(j_i\cdots j_N)$ are two permurations of $(1\cdots N)$ with $i_n\neq j_n$, $1\leqslant n\leqslant N$. Their contribution is

$$\int_\rho^1 \frac{dx_{2N}}{x_{2N}^2}\cdots\int_\rho^{x_{N+2}}\frac{dx_{N+1}}{x_{N+1}^2}\int_\rho^{x_{N+1}} dx_N\cdots\int_\rho^{x_2} dx_1 \sum_{n=0}^{\infty}\left(\frac{x_1\cdots x_N}{x_{N+1}\cdots x_{2N}}\right)^{2n}$$

i.e., in the limit $\rho\to 0$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\,(4n+2)^2\cdots[2(N-1)\,n+N-1]^2\,(2Nn+N)}\int_\rho^1\frac{dx_{2N}}{x_{2N}}$$

We thus get

$$C_{2N}(1/2)=(-1)^{N-1}\,(2N-2)!\,\frac{N}{(N!)^2}\,\zeta(2N-1)(1-2^{1-2N})$$

or

$$C_{2N}(1/2)=2^{2N-1}(-1)^N\frac{\Gamma(N-1/2)}{\Gamma(-1/2)\,N!}\,\zeta(2N-1)(1-2^{1-2N}) \qquad (4.19)$$

This agrees with the expansion in powers of t of the varying "central charge" deduced from (3.3),

$$\begin{aligned}C(t)=&\frac{1}{24}-\frac{t^2}{2}\ln\left[\pi e^{-\gamma}\left(\frac{\operatorname{Im}\tau}{A}\right)^{1/2}\right]\\&+\sum_{N=2}^{\infty}2^{2N-1}(-1)^N\frac{\Gamma(N-1/2)}{\Gamma(-1/2)\,N!}\,\zeta(2N-1)(1-2^{1-2N})\,t^{2N} \qquad (4.20)\end{aligned}$$

3. We turn now to the magnetic perturbation in the Ising model, φ being the spin operator σ with dimensions (1/16, 1/16). Correlation functions are given by[8]

$$\begin{aligned}&\langle 0|\,\sigma(x_1,\bar{x}_1)\cdots\sigma(x_{2N},\bar{x}_{2N})\,|0\rangle^2\\&\quad=\left(\frac{1}{2}\right)^{N/2}\sum_{\substack{\varepsilon_i=\pm 1\\ \sum\varepsilon_i=0}}\ \prod_{1\leqslant i<j\leqslant 2N}|x_{ij}|^{\varepsilon_i\varepsilon_j/2} \qquad (4.21)\end{aligned}$$

In this case it is the square of the correlation function that has a simple form, and taking the square root leads to more complicated calculations than above. We shall consider only $N=2$. We can write

$$\langle 0|\, \sigma(1)\cdots\sigma(4)\, |0\rangle$$
$$= \frac{1}{(x_{14}x_{23}u)^{1/8}} \frac{1}{(\bar{x}_{14}\bar{x}_{23}\bar{u})^{1/8}} \{|f_+(u)|^2 + |f_-(u)|^2\} \tag{4.22}$$

where

$$f_\pm(u) = \left[\frac{1 \pm (1-u)^{1/2}}{2}\right]^{1/2}, \qquad u = \frac{x_{12}x_{43}}{x_{13}x_{42}}$$

Both functions $f_\pm(u)$ satisfy the differential equation

$$u(1-u)\, f'' + (\tfrac{1}{2} - u)\, f' + \tfrac{1}{16} f = 0 \tag{4.23}$$

which enables one to expand them in power of u in the case of f_+ and of $u^{1/2}$ in the case of f_-, $|u| \leqslant 1$:

$$\begin{aligned} f_+(u) &= \left[\frac{1 + (1-u)^{1/2}}{2}\right]^{1/2} = \sum_{m=0}^{\infty} A_m u^m \\ f_-(u) &= \left[\frac{1 - (1-u)^{1/2}}{2}\right]^{1/2} = -\sum_{m=0}^{\infty} A_{m+1/2} u^{m+1/2} \end{aligned} \tag{4.24}$$

with

$$A_x = \frac{\Gamma(1/2)}{\Gamma(1/4)\,\Gamma(-1/4)} \frac{\Gamma(x+1/4)\,\Gamma(x-1/4)}{\Gamma(x+1)\,\Gamma(x+1/2)}$$

One can then develop (4.22) for radialy ordered variables. After long but straightforward algebra and repeated use of the binomial formula, one gets a sum over 14 indices,

$$\langle 0|\, \sigma(1)\cdots\sigma(4)\, |0\rangle$$
$$= \sum_{\substack{n_1\cdots n_6, m,\\ \bar{n}_1\cdots\bar{n}_6, \bar{m} = 0}}^{\infty} \prod_{i=1}^{2} \frac{\Gamma(n_i + 1/8)}{\Gamma(1/8)\, n_i!} \prod_{i=3}^{4} \frac{\Gamma(n_i + 1/8 - m)}{\Gamma(1/8 - m)\, n_i!} \prod_{i=5}^{6} \frac{\Gamma(n_i + m - 1/18)}{\Gamma(m - 1/8)\, n_i!} A_m$$
$$\times (x_2 x_4)^{-1/8} \left(\frac{x_1}{x_2}\right)^{n_1+n_3+n_5} \left(\frac{x_2}{x_3}\right)^{n_1+n_2+n_5+n_6+m} \left(\frac{x_3}{x_4}\right)^{n_1+n_4+n_6}$$
$$\times \text{C.C.} + \text{same sum with } m(\bar{m}) \text{ replaced by } m + 1/2\ (\bar{m} + 1/2) \tag{4.25}$$

where complex conjugation means also use of $\bar{n}$ indices instead of n. Although (4.25) was obtained using the expansion (4.24), legitimate for $|u| \leqslant 1$ only, it now defines a series that is convergent for radially ordered variables, and we expect this expression to be true whatever the value of $|u|$. To get the connected part, one must subtract

$$\langle 0|\, \sigma(1)\, \sigma(2)\, |0\rangle \langle 0|\, \sigma(3)\, \sigma(4)\, |0\rangle$$

which can be expanded as

$$(x_2 x_4)^{-1/8} \sum_{P_1 P_2 = 0}^{\infty} \prod_{i=1}^{2} \frac{\Gamma(P_i + 1/8)}{\Gamma(1/8)\, P_i!} \left(\frac{x_1}{x_2}\right)^{P_1} \left(\frac{x_3}{x_4}\right)^{P_2} \times \text{C.C.} \tag{4.26}$$

and two similar terms deduced by permutation,

$$(x_3 x_4)^{-1/8} \sum_{P_1 P_2 = 0}^{\infty} \prod_{i=1}^{2} \frac{\Gamma(P_i + 1/8)}{\Gamma(1/8)\, P_i!} \left(\frac{x_1}{x_3}\right)^{P_1} \left(\frac{x_2}{x_4}\right)^{P_2} \times \text{C.C.} \tag{4.27}$$

$$(x_3 x_4)^{-1/8} \sum_{P_1 P_2 = 0}^{\infty} \prod_{i=1}^{2} \frac{\Gamma(P_i + 1/8)}{\Gamma(1/8)\, P_i!} \left(\frac{x_2}{x_3}\right)^{P_1} \left(\frac{x_1}{x_4}\right)^{P_2} \times \text{C.C.} \tag{4.28}$$

We now perform the integration (4.12). For a term like

$$\frac{1}{(2\pi)^{2N}} \int_{\rho \leqslant |x_1| \cdots \leqslant |x_{2N}| \leqslant 1} \frac{d^2 x_1 \cdots d^2 x_{2N}}{(x_1 \bar{x}_1 \cdots x_{2N} \bar{x}_{2N})^{15/16}}$$

$$\times (x_2 x_4)^{-1/8} \left(\frac{x_1}{x_2}\right)^{n_1 + n_3 + n_5} \left(\frac{x_2}{x_3}\right)^{n_1 + n_2 + n_5 + n_6 + m} \left(\frac{x_3}{x_4}\right)^{n_1 + n_4 + n_6} \times \text{C.C.}$$

the angular part selects terms where $n_1 + n_3 + n_5 = \bar{n}_1 + \bar{n}_3 + \bar{n}_5$ and similar relations for the two other exponents, with the leading behavior as $\rho \to 0$

$$\frac{1}{1/8 + 2(n_1 + n_3 + n_5)} \frac{1}{2(n_1 + n_2 + n_5 + n_6 + m)} \frac{1}{1/8 + 2(n_1 + n_4 + n_6)} \int_\rho^1 \frac{dx_4}{x_4}$$

The case where the second denominator vanishes is excluded by the subtraction of (4.26). Integration of (4.27) or (4.28) is similar to the calculation in (4.13), (4.14). Collecting results, we get finally

$$
\begin{aligned}
C_4\left(\frac{1}{16}\right) = & \sum_{\substack{n_1 \cdots n_6, m, \\ \bar{n}_1 \cdots \bar{n}_6, \bar{m} = 0 \\ n_1 + n_3 + n_5 = \text{C.C.} \\ n_1 + n_2 + n_5 + n_6 + m = \text{C.C.} \neq 0 \\ n_1 + n_4 + n_6 = \text{C.C.}}} \left(\prod_{i=1}^{2} \frac{\Gamma(n_i + 1/8)}{\Gamma(1/8)\, n_i!} \right. \\
& \left. \times \prod_{i=3}^{4} \frac{\Gamma(n_i + 1/8 - m)}{\Gamma(1/8 - m)\, n_i!} \prod_{i=5}^{6} \frac{\Gamma(n_i + m - 1/8)}{\Gamma(m - 1/8)\, n_i!} A_m \times \text{C.C.} \right) \\
& \times \left(\frac{1}{1/8 + 2(n_1 + n_3 + n_5)} \frac{1}{2(n_1 + n_2 + n_5 + n_6 + m)} \right. \\
& \left. \times \frac{1}{1/8 + 2(n_1 + n_4 + n_6)} \right) \\
& + \text{same sum with } m\ (\bar{m}) \to m + 1/2\ (\bar{m} + 1/2) \\
& - \sum_{n_7, n_8 = 0}^{\infty} \left(\prod_{i=7}^{8} \frac{\Gamma(n_i + 1/8)}{\Gamma(1/8)\, n_i!} \times \text{C.C.} \right) \\
& \times \left(\frac{1}{1/8 + 2n_7} \frac{1}{1/8 + 2n_8} \frac{1}{1/4 + 2(n_7 + n_8)} \right. \\
& \left. + \frac{1}{(1/8 + 2n_7)^2} \frac{1}{1/4 + 2(n_7 + n_8)} \right)
\end{aligned} \tag{4.29}
$$

In contrast to $C_4(1/2)$, this expression is rather heavy, and does not seem to generalize simply to higher orders. In the strip limit ($T \to \infty$) it is interesting to consider the first cumulants of the magnetization M for a discrete Ising model mode of $LT = A$ spins at criticality,

$$
\begin{aligned}
m_2(L) &= \lim_{T \to \infty} A^{-1} \langle M^2 \rangle \\
m_4(L) &= \lim_{T \to \infty} A^{-1} [\langle M^4 \rangle - 3 \langle M^2 \rangle^2]
\end{aligned} \tag{4.30}
$$

The ratio

$$
R = -\frac{1}{3} \lim_{L \to \infty} \frac{m_4(L)}{[L m_2(L)]^2} \tag{4.31}
$$

is universal and has been evaluated numerically,

$$
R = 2.46044 \pm 0.00002 \tag{4.32}
$$

using transfer matrix methods[9] or a Monte Carlo estimation[10] of the integral (4.12). We get here its analytic form

$$R = -\frac{C_4(1/16)}{\pi[C_2(1/16)]^2} \tag{4.33}$$

C_2 is easily obtained numerically, $C_2 \simeq 8.009487$.

The calculation of C_4 is more difficult, since the sum involves 11 indices. Fortunately, it converges quite rapidly. (For all n_i, m on the same scale n, Stirling's formula $\Gamma(n) \sim n^{n-1/2}e^{-n}$ gives a behavior $\sim n^{-23/4}$) and the first few terms provide a good estimation. Computation for n_i, $m \in [0, 3]$ gives $C_4 \simeq -495.88 \pm 0.01$. Thus,

$$R = 2.46048 \pm 0.00005 \tag{4.34}$$

in agreement with (4.32).

Note that the zeroth-order term in (4.14) is 8 and $0 + 16 - 512 = -496$ in (4.29), leading to an estimate $R \simeq -31/4\pi = -2.467$ already very close to (4.34).

4. One can also perform similar calculations for the varying "dimensions." For the spin operator in the Ising case one has, for instance,

$$\begin{aligned} Z_{1/16}(g, \operatorname{Im}\tau) &= \langle\sigma|\exp(-\operatorname{Im}\omega_2 H + iP\operatorname{Re}\omega_2)|\sigma\rangle \\ &= \exp\frac{-2\pi\operatorname{Im}\tau}{12}\sum_{N=0}^{\infty}\left(\frac{g}{2\pi}\right)^N \\ &\quad\times\int_{\rho\leqslant|x_1|\cdots\leqslant|x_N|\leqslant 1}\frac{d^2x_1\cdots d^2x_N}{(x_1\bar{x}_1\cdots x_N\bar{x}_N)^{1-h}}\frac{d\theta}{2\pi} \\ &\quad\times\langle 0|\sigma(0,0)\,\varphi(x_1,\bar{x}_1)\cdots\varphi(x_N,\bar{x}_N) \\ &\quad\times\sigma(\exp i\theta, \exp -i\theta)|0\rangle \end{aligned} \tag{4.35}$$

which defines

$$D(g) = \lim_{\rho\to 0}\frac{\ln Z_{1/16}(g,\rho)}{\ln(1/\rho)} \tag{4.36}$$

with

$$D(g) = -1/12 + gD_1 + g^2D_2 + \cdots \tag{4.37}$$

For the thermal perturbation the first term is nonzero. We must compute

$$\frac{1}{2\pi}\int_{\rho\leqslant|x_2|\leqslant 1}\frac{d^2x_2}{(x_2\bar{x}_2)^{1/2}}\frac{d\theta_3}{2\pi}\langle 0|\sigma(0,0)\,\epsilon(x_2,\bar{x}_2)\,\sigma(e^{i\theta_3}, e^{-i\theta_3})|0\rangle$$

This integral requires knowledge of the three-point function, which is easily deduced from (4.13), (4.21):

$$\langle 0|\, \sigma(x_1\bar{x}_1)\, \varepsilon\, (x_2\bar{x}_2)\, \sigma(x_3\bar{x}_3)\, |0\rangle = \tfrac{1}{2} x_{12}^{-1/2} x_{23}^{-1/2} x_{31}^{3/8} \times \text{C.C.} \tag{4.38}$$

We are thus left with

$$\frac{1}{4\pi} \int_{\rho \leqslant |x_2| \leqslant 1} \frac{d^2x_2}{(x_2\bar{x}_2)^{1/2}} \frac{d\theta_3}{2\pi} (x_2\bar{x}_2)^{-1/2} (e^{i\theta_3} - x_2)^{-1/2} (e^{-i\theta_3} - \bar{x}_2)^{-1/2}$$

whose dominant behavior is simply $1/2 \ln(1/\rho)$. Thus,

$$D_1(1/2) = 1/2 \tag{4.39}$$

in agreement with (3.3). Note that a similar calculation of the "varying dimension" of the energy (identity) would give zero, since $\langle \varepsilon\varepsilon\varepsilon \rangle$ ($\langle I \varepsilon I \rangle$) vanishes. As an example of a new result, we consider the magnetic perturbation. Then $D_1(1/16) = 0$ and one obtains D_2 using the correlation function (4.25)

$$\begin{aligned} D_2\left(\frac{1}{16}\right) = \sum_{n_1 n_2 m = 0}^{\infty} & \frac{\Gamma(n_1 + 1/8)}{\Gamma(1/8)\, n_1!} \frac{\Gamma(n_2 + 1/8)}{\Gamma(1/8)\, n_2!} A_m A_{u_1 - n_2 + m} \\ & \times \frac{1}{2(n_1 + m) - 1/8} \\ & + \text{same sum with } m \to m + 1/2 \end{aligned} \tag{4.40}$$

We thus predict the ratio of the shift in g^2 for the ground-state and the first excited-state energies

$$r = \frac{C_2(1/16)}{D_2(1/16)} = -1.03926 \tag{4.41}$$

where C_2, D_2 have been computed numerically using the above expressions [note that the zeroth-order approximation to the sums (4.14) and (4.40) already gives $r \simeq -1$]. Transfer matrix estimations of this quantity, obtained by a method similar to the one used in Ref. 9, are given in Table 1. Their extrapolation agrees satisfactorily with the value given by (4.41).

5. All these calculations are by no means restricted to the Ising case. As a final example, we consider the three-state Potts model with a thermal

Table I. Estimates of the Ratio r [Eq. (4.41)][a]

L	r
1	-1
2	-1.027820
3	-1.033255
4	-1.035283
5	-1.036376
6	-1.037065
7	-1.037530
8	-1.037858
9	-1.038099
10	-1.038232
Extrapolated value	-1.0392 ± 0.0003

[a] These values have been obtained by considering a discrete Ising model on the square lattice with a cylinder geometry (L being evaluated in lattice spacing units). We have computed numerically the shift in H^2 of the ground-state and first excited-state energies in the transfer matrix spectrum.

perturbation $\mathscr{E}(2/5, 2/5)$. Then C_2 is given by (4.14) and one can obtain C_4 using the known four-point energy correlation function[11]

$$
\begin{aligned}
&\langle 0|\, \mathscr{E}(1)\, \mathscr{E}(2)\, \mathscr{E}(3)\, \mathscr{E}(4) \rangle \\
&\quad = (x_{14} x_{23} u)^{-4/5}\, F(-8/15,\ -1/5;\ -2/5; u) \times \text{C.C.} \\
&\quad\quad - \frac{\Gamma^2(-2/5)\, \Gamma(6/5)\, \Gamma(13/5)}{\Gamma(-8/5)\, \Gamma(-1/5)\, \Gamma^2(12/5)} (x_{14} x_{23} u)^{3/5} (x_{13} x_{24})^{-7/5} \\
&\quad\quad \times F(13/5,\ 6/5;\ 12/5; u) \times \text{C.C.}
\end{aligned}
\tag{4.42}
$$

where F is the Gauss series

$$
F(a, b; c; u) = \sum_{m=0}^{\infty} A_m(a, b; c)\, u^m \tag{4.43a}
$$

with

$$
A_m(a, b; c) = \frac{\Gamma(a+m)\, \Gamma(b+m)\, \Gamma(c)}{\Gamma(a)\, \Gamma(b)\, \Gamma(c+m)\, m!} \tag{4.43b}
$$

The calculation is similar to the preceding ones and we only quote the result here:

$$C_4\left(\frac{2}{5}\right) = \sum_{\substack{n_1\cdots n_6, m,\\ \bar n_1\cdots \bar n_6, \bar m = 0\\ n_1+n_3+n_5=\mathrm{C.C.}\\ n_1+n_2+n_5+n_6+m=\mathrm{C.C.}\neq 0\\ n_1+n_4+n_6=\mathrm{C.C.}}} \left(\prod_{i=1}^{2}\frac{\Gamma(n_1+4/5)}{\Gamma(4/5)\,n_i!}\right.$$

$$\times \prod_{i=3}^{4}\frac{\Gamma(n_i+4/5-m)}{\Gamma(4/5-m)\,n_i!}\prod_{i=5}^{6}\frac{\Gamma(n_i+m-4/5)}{\Gamma(m-4/5)\,n_i!}\Lambda_m^{(1)}\times \mathrm{C.C.}\left.\right)$$

$$\times\left(\frac{1}{4/5+2(n_1+n_3+n_5)}\frac{1}{2(n_1+n_2+n_5+n_6+m)}\right.$$

$$\left.\times\frac{1}{4/5+2(n_1+n_4+n_6)}\right)$$

$$-\frac{\Gamma^2(-2/5)\,\Gamma(6/5)\,\Gamma(13/5)}{\Gamma(-8/5)\,\Gamma(-1/5)\,\Gamma^2(12/5)}$$

$$\sum_{\substack{n_1\cdots n_8, m,\\ \bar n_1\cdots \bar n_8, \bar m = 0\\ n_1+n_3+n_5+n_7=\mathrm{C.C.}\\ n_1+n_2+n_5+n_6+n_7+n_8+m=\mathrm{C.C.}\\ n_1+n_4+n_6+n_8=\mathrm{C.C.}}}^{\infty} \left(\prod_{i=1}^{2}\frac{\Gamma(n_i-3/5)}{\Gamma(-3/5)\,n_i!}\prod_{i=3}^{4}\frac{\Gamma(n_i-m-3/5)}{\Gamma(-m-3/5)\,n_i!}\right.$$

$$\times \prod_{i=5}^{6}\frac{\Gamma(n_i+m+3/5)}{\Gamma(m+3/5)\,n_i!}\prod_{i=7}^{8}\frac{\Gamma(n_i+7/5)}{\Gamma(7/5)\,n_i!}\Lambda_m^{(2)}\times \mathrm{C.C.}\left.\right)$$

$$\times\left(\frac{1}{4/5+2(n_1+n_3+n_5+n_7)}\right.$$

$$\times\frac{1}{14/5+2(n_1+n_2+n_5+n_6+n_7+n_8+m)}$$

$$\left.\times\frac{1}{4/5+2(n_1+n_4+n_6+n_8)}\right)$$

$$-\sum_{n_9,n_{10}=0}^{\infty}\left(\prod_{i=9}^{10}\frac{\Gamma(n_i+4/5)}{\Gamma(4/5)\,n_i!}\times \mathrm{C.C.}\right)$$

$$\times\left(\frac{1}{4/5+2n_9}\frac{1}{4/5+2n_{10}}\frac{1}{8/5+2(n_9+n_{10})}\right.$$

$$\left.+\frac{1}{(4/5+2n_9)^2}\frac{1}{8/5+2(n_9+n_{10})}\right) \tag{4.44}$$

where $\Lambda_m^{(1)}$ and $\Lambda_m^{(2)}$ are the coefficients (4.43b) in the expansions of the two Gauss series (4.43a).

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